ELECTRIC-ANALOG SOLUTION OF PROBLEMS OF NONSTEADY HEAT CONDUCTION WITH TIME-DEPENDENT BOUNDARY CONDITIONS OF THE THIRD KIND

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The possibility of substituting piecewise-linear for continuous $\alpha = f(\tau)$ laws has been experimentally investigated in connection with the electric-analog solution of problems of nonsteady heat conduction with time-dependent boundary conditions of the third kind.

The investigation of the nonsteady thermal states of heat engine components involves the solution of the Fourier equation

$$\frac{\partial}{\partial x}\left[\lambda \frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[\lambda \frac{\partial T}{\partial y}\right]+\frac{\partial}{\partial z}\left[\lambda \frac{\partial T}{\partial z}\right]=c\gamma \frac{\partial T}{\partial \tau} \qquad (1)$$

with boundary conditions of the third kind

$$\alpha(x,\ y,\ z,\ \tau)\,[T_m(x,\ y,\ z,\ \tau)-T_{con}(x,\ y,\ z,\ \tau)]=\lambda \frac{\partial T}{\partial n}. \qquad (2)$$

Although, in transient regimes of heat engine operation, not only the temperature of the medium but also the heat-transfer coefficients vary within fairly wide limits, modern analog computers, which can be used to solve problems of nonsteady heat conduction in bodies of complex configuration, do not have means of assigning time-dependent boundary conditions of the third kind with allowance for the variability of both the heat-transfer coefficients $\alpha = f(\tau)$ and the temperature $T_m = f(\tau)$. Only some of them [1, 2] have means of simulating $T_m = f(\tau)$.

This situation has led to various attempts to take the variability of the heat-transfer coefficients into account. For example, Ivashchenko and Zolotogorov [3] propose a method of successive approximations consisting essentially in leaving one component (α) constant, while the other (T_m) is allowed to vary, the $T_m = f(\tau)$ relation being corrected starting from the conditions of conservation of the true heat fluxes at the boundaries of the body. Although this method gives acceptable accuracy after a relatively small number of approximations, it is really suitable only for systems with a limited number of nodes. In the presence of a considerable number of boundary nodes (components of complex configuration), in solving problems of nonsteady heat conduction on RC networks the continuous function $\alpha = f(\tau)$ is replaced with a piecewise-linear one, with the α constant on a bounded time interval and changing stepwise on the next interval [2].

It now becomes necessary to determine the maximum errors introduced into the nonsteady temperature field by substituting α = const for α = var on a finite number of intervals. In view of the lack of information of this kind, it is necessary to reduce the time intervals of constant α, which involves a substantial increase in the number of reassignments of the initial and boundary conditions.

In solving problems of nonsteady heat conduction by the Liebmann method on R networks [4] and choosing the time step with allowance for the variation of $T_m = f(\tau)$, the lack of information on the degree of influence of $\alpha = f(\tau)$ on the temperature field also leads to the necessity of reducing the time step and adding to the time required to set up the problem, if the intervals of constant α are shorter than the intervals of constant T_m.

The authors' investigation of the accuracy of assignment of the boundary resistances in the electric-analog solution of certain problems of nonsteady heat conduction provides only a partial answer to the question of the error introduced into the temperature field when α = const is substituted for α = var over a broad range of variation of the heat-transfer coefficients. In this investigation, we examined only the effect of the error in assigning the boundary conditions on the variation of the temperature at individual points of bodies of complex configuration for limited deviations of α. It was therefore necessary to broaden the scope of the investigation to answer the question of the permissibility of a piecewise-linear approximation of the function $\alpha = f(\tau)$ in problems of nonsteady heat conduction with a finite number of intervals of constancy.

The investigations were carried out on the R network of the MSM-1 analog computer, modernized for solving nonsteady problems of heat conduction [5], and on a USM-1. As objects of investigation we selected the rotor and housing of a steam turbine, at whose boundaries very considerable changes in the rate of heat transfer are observed during startup.

Since our aim reduced to determining the distortion of the temperature field caused by substituting a piecewise-linear for a continuous law of variation $\alpha = f(\tau)$, to exclude the influence of other factors in the R-network solutions we considered only problems in which the temperature of the medium was constant. The presence of a functional converter unit in the USM-1 made it possible to solve on RC networks problems in which the variability of $T_m = f(\tau)$ is taken into account, without further distorting the temperature field as a result of an error in T_m, or, at any rate, with such distortion reduced to a minimum.

In selecting the piecewise-linear function $\alpha_n = f(\tau)$ which approximates the function $\alpha = f(\tau)$, as the approximate value of α on each time interval $[\tau_i, \tau_{i+1}]$, at whose boundaries the values of the function are, respectively, equal to α_i and α_{i+1}, we took a value α_{a_i} whose deviation from the actual function did not exceed an amount δ_{max} at the ends of each time interval:

$$\alpha_{a_i} = \left[1 + \frac{\delta_{max}}{100}\right] \alpha_i = \left[1 - \frac{\delta_{max}}{100}\right] \alpha_{i+1}. \qquad (3)$$

These results showed that the piecewise-linear function $\alpha_n = f(\tau)$ approximates the function $\alpha = f(\tau)$.

In the first cycle of investigation, the laws of piecewise-linear approximation were selected so that on no time interval did δ_{max} exceed ±5%, ±10%, or ±15%.

The curves in Fig. 1b show the variation of the relative temperature at certain points of the rotor and turbine housing in the heating regime for variation of the heat-transfer coefficient at the surface according to the law of Fig. 1a. These curves convincingly show that substituting for the continuous law $\alpha = f(\tau)$ a piecewise-linear one with $\delta_{max} \leq \pm 15\%$ has practically no effect on the temperature variation even at points on the heat-transfer surface.

Considering that we covered quite a broad range of variation of the ratio of the internal and external thermal resistances (α/λ = 1–20) and that in solving on R networks a decrease in the departures of the approximate values α_a from the real values was simultaneously accompanied by a decrease in the time step $\Delta\tau$ and, hence, R_t, it may be assumed that in solving problems of nonsteady heat conduction with time-dependent boundary conditions of the third kind on electric analogs it is possible to substitute piecewise-linear for continuous $\alpha = f(\tau)$ laws. The solution obtained will be accurate enough if $\delta_{max} \leq \pm 15\%$.

By replacing the continuous $\alpha = f(\tau)$ law with a piecewise-linear one with a maximum deviation δ_{max} = = ±15% on each time interval, it is possible, if the time of the process is divided into ten intervals, to obtain an almost exact solution in problems in which the heat-transfer coefficients vary by a factor of up to 20 during the process. Hence, in using RC networks to solve problems of nonsteady heat conduction with time-dependent boundary conditions of the third kind, given a piecewise-linear approximation of the functions $\alpha = f(\tau)$, the necessary reassignment of the boundary resistances at a point is limited.

However, since in transient regimes the heat-transfer coefficients at different points vary at different rates, it may turn out that the reassignment of the boundary resistances at the individual points is displaced in time, so that the RC network solution is no different from the R network solution, since the time intervals in the reassignment of the initial and boundary conditions remain sufficiently small.

Of course, it is necessary to estimate the error introduced into the temperature field on the assumption of much larger δ_{max}. Accordingly, we used the same elements in an investigation to determine the distortion of the temperature field in the presence of a rougher approximation in the assignment of the boundary conditions.

As the results show (Fig. 2), for δ_{max} on the order of ±30%, the maximum error in the relative temperature at the heat-transfer surface does not exceed

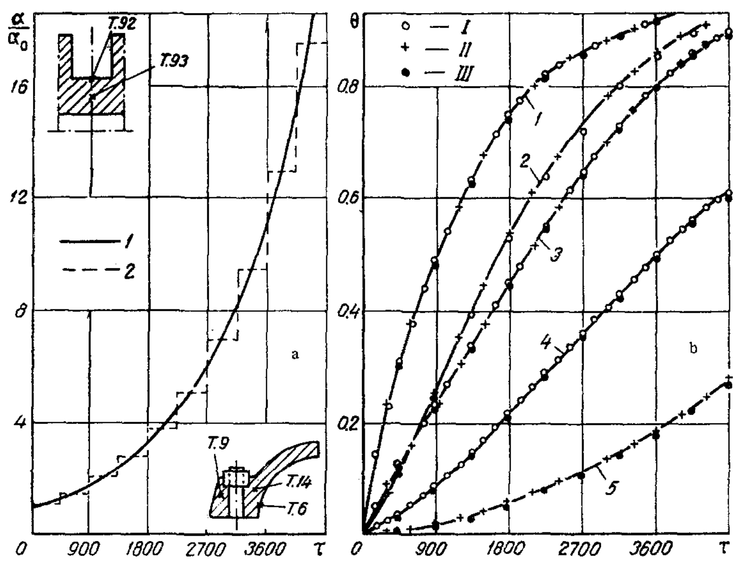

Fig. 1. Variation of the rate of heat transfer (a) at the boundaries of the investigated region $\alpha/\alpha_0 = f(\tau)$ at α_0 = 29 W/m² · deg for the housing and α_0 = 58.3 W/m² · deg for the rotor of a steam turbine: 1) calculation; 2) piecewise-linear approximation with δ_{max} = ±15%. Variation of the relative temperature (b) at individual points of the elements of the turbine housing and rotor for $\alpha/\alpha_0 = f(\tau)$ in accordance with Fig. 1a: 1) at point 92; 2) at point 93; 3) at point 6; 4) at point 14; 5) at point 9; I) for δ_{max} = ±5%; II) for δ_{max} = ±10%; III) for δ_{max} = ±15%.

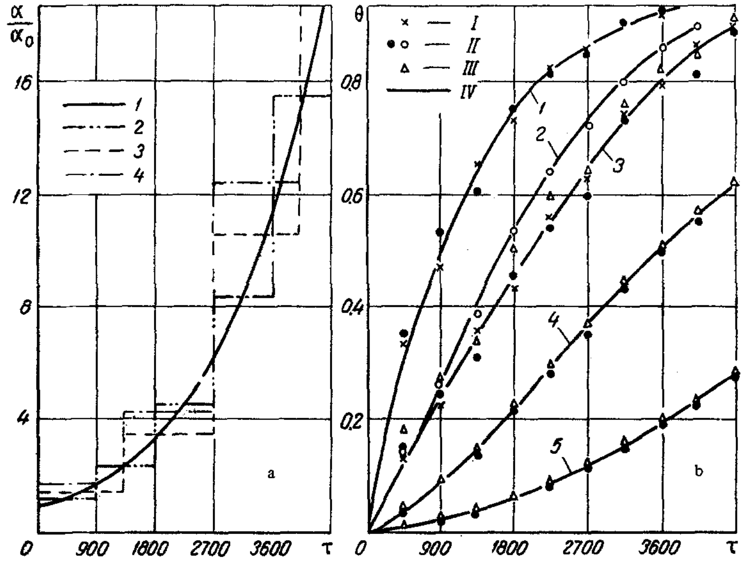

Fig. 2. Variation of the rate of heat transfer (a) at the boundaries of the investigated region $\alpha/\alpha_0 = f(\tau)$ at α_0 = 29 W/m^2 · deg for the housing and α_0 = 58.3 W/m^2 · deg for the rotor of a steam turbine: 1) calculation; 2) piecewise-linear approximation with δ_{max} = ±30%; 3) the same with δ_{max} = ±42.5%; 4) the same when the mean value of the function on each time interval is taken as the approximating value, δ_{max} = +74%, −30%. Variation of the relative temperature (b) at individual points of the turbine housing and rotor (Fig. 1a) for variation of $\alpha/\alpha_0 = f(\tau)$ in accordance with Fig. 2a: 1) at point 92; 2) at point 93; 3) at point 6; 4) at point 14; 5) at point 9; I) for δ_{max} = ±30%, II) for δ_{max} = ±42.5%; III) for δ_{max} = +74%, −30%; IV) calculated curve for δ_{max} = ±15%.

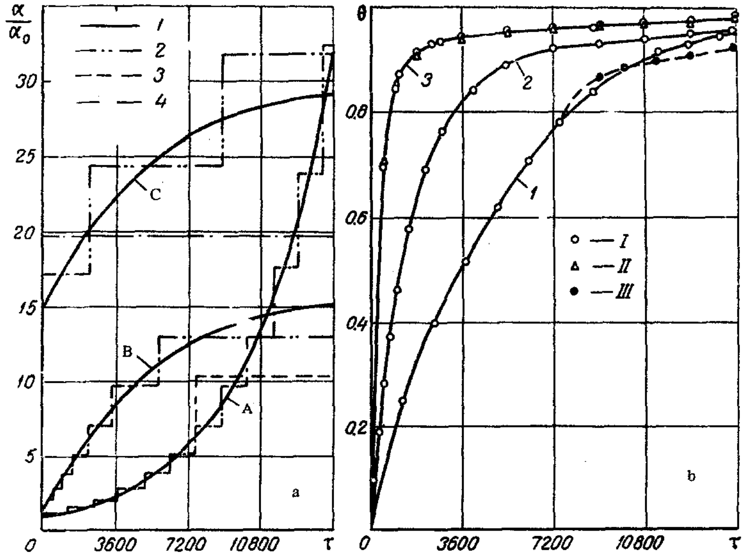

Fig. 3. Variation of the rate of heat transfer (a) at the boundaries of the investigated region $\alpha/\alpha_0 = f(\tau)$ at α_0 = 29 W/m^2 · deg: 1) calculation; 2) piecewise-linear approximation with δ_{max} = = ±15%; 3) the same with δ_{max} = ±32%; 4) the same with δ_{max} = = ±70%. Variation of the relative temperature (b) at the point 6 of the steam-turbine housing (Fig. 1a) for various laws of variation of $\alpha/\alpha_0 = f(\tau)$: 1) for law A; 2) for law B; 3) for law C (Fig. 3a); I) for δ_{max} = ± 15%; II) for δ_{max} = ±32%; III) for δ_{max} = ±70%.

±2.5%. It should be noted that for points remote from the heat-transfer surface this error is sharply reduced.

At δ_{max} = ±42.5% the maximum error in the temperature at the heat-transfer surface does not exceed ±4.5%.

In analyzing data on the variation of the relative temperature $\bar{\theta} = f(\mathrm{Bi}, \tau)$ at individual points of a turbine rotor [6] and data on the variation of $\bar{\theta} = f(\mathrm{Bi}, \tau)$ in other elements, it is necessary to emphasize that when a certain level of heat transfer, corresponding to a ratio $\alpha/\lambda \geq 8$, is reached it is possible to accept a much greater departure of the approximate values of the heat-transfer coefficients α_a from the real values without any significant distortion of the temperature distribution.

In Fig. 3b the curves showing the variation of the relative temperature at point 6 of the turbine housing (see Fig. 1a) are presented for various laws $\alpha = f(\tau)$ and various methods of piecewise-linear approximation.

The curves show that at large ratios α/λ it is also possible to accept a large error in assigning α without seriously distorting the temperature field. For example, in the case of curve A in Fig. 3a from the moment corresponding to a ratio $\alpha/\lambda > 7.0$, δ_{max} was equal to 70%, but the error of the relative temperature did not exceed ±4.5%. At high values of α ($\alpha/\lambda > 15$) replacing the real function $\alpha = f(\tau)$ with a piecewise-linear one with δ_{max} = ±15% and α = const with δ_{max} = ±32% (curve 3 in Fig. 3b) has almost no effect on the variation of the relative temperature with time at the point in question.

The results of the last cycle of investigation show that when a continuous function is replaced with a piecewise-linear one in problems with a large variation of the heat-transfer coefficients at the boundaries, it is desirable to make δ_{max} variable, increasing it as the ratio α/λ increases. This makes it possible to replace any function $\alpha = f(\tau)$ with a piecewise-linear function with up to ten intervals of constant α while preserving the accuracy of the solution on electric analogs.

Thus, our results permit one to select, from the given laws of variation of the heat-transfer coefficients at the boundaries of the investigated regions, the duration of the time intervals for which the variability of α can be neglected, and to determine the errors in the temperature field resulting from the substitution of piecewise-linear for continuous $\alpha = f(\tau)$ laws.

It should be kept in mind that, since in solving on R networks the function $T_m = f(\tau)$ is also expressed as a piecewise-linear approximation, the durations of the intervals of constant T_m and α must be coordinated. In view of the decisive influence of T_m on the temperature field [4], it is desirable to make the intervalues of constant α equal to, or multiples of, the intervals of constant T_m.

On the basis of these investigations we developed a controlled-resistance unit (CRU) for simulating time-dependent heat-transfer coefficients. As the α-simulating elements of this unit we took boundary resistance circuits (Fig. 4) controlled by pulses from the timer of the USM-1 functional converter unit. The use of a CRU makes possible the continuous solution of problems of nonsteady heat conduction with time-dependent boundary conditions of the third kind on RC networks.

As a result of the presence in USM-1 type machines of a unit for simulating the time-dependent temperature $T_m = f(\tau)$ of the medium, in this case the error in the temperature field of the investigated region is determined by the error introduced by the piecewise-linear approximation of the function $\alpha = f(\tau)$ in the CRU.

To calculate the boundary resistance circuits, we developed a method that makes it possible for a finite number of intervals of constant α to obtain a minimum error in the solution of the problem. This method can also be used for calculating the time intervals (τ_j) on which the variability of the heat-transfer coefficients can be neglected without loss of accuracy and the assigned values of the boundary resistances on these intervals (R_{bi}) in solving problems of nonsteady heat conduction by the Liebmann method on R networks.

It can be shown that when any function $\alpha = f(\tau)$ is replaced with a piecewise-linear function $\alpha_n = f(\tau)$ for each selected δ_{max}, irrespective of the form of the function $\alpha = f(\tau)$, the relative values $\bar{\alpha}_{ai} = \alpha_{ai}/\alpha_0$ and $\bar{\alpha}_{i+1} = \alpha_{i+1}/\alpha_0$, at which the approximate values of the heat-transfer coefficients change stepwise from $\bar{\alpha}_{ai}$ to $\bar{\alpha}_{ai+1}$, are strictly defined and re-

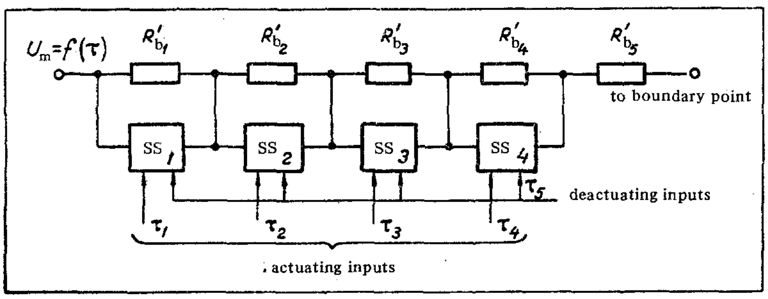

Fig. 4. Boundary resistance circuit simulating time-dependent heat-transfer coefficients: SS denotes the semiconductor switches effecting a stepwise change in the internal resistance of the circuit, τ_i the time pulses controlling the operation of the switches; $U_m = f(\tau)$ is a function simulating the variation of the temperature of the medium with time.

lated by the expressions:

$$\bar{\alpha}_{a_1} = \left(1+\frac{\delta_{max}}{100}\right)\bar{\alpha}_1 = \left(1+\frac{\delta_{max}}{100}\right), \quad \text{since } \bar{\alpha}_1 = 1, \tag{4}$$

$$\bar{\alpha}_{a_1} = \left(1-\frac{\delta_{max}}{100}\right)\bar{\alpha}_2, \tag{5}$$

$$\bar{\alpha}_2 = \frac{1+\frac{\delta_{max}}{100}}{1-\frac{\delta_{max}}{100}}, \tag{6}$$

. . . .

$$\bar{\alpha}_i = \left(\frac{1+\frac{\delta_{max}}{100}}{1-\frac{\delta_{max}}{100}}\right)^{i-1}, \tag{7}$$

$$\bar{\alpha}_{a_i} = \bar{\alpha}_i\left(1+\frac{\delta_{max}}{100}\right), \tag{8}$$

$$\delta_{max} = \left(\frac{\frac{\alpha_{i+1}}{\alpha_i}-1}{\frac{\alpha_{i+1}}{\alpha_i}+1}\right)\cdot 100. \tag{9}$$

Here, i varies from 1 to n + 1 for $\bar{\alpha}_i$ and from 1 to n for $\bar{\alpha}_{ai}$.

If we assign the number of intervals of constancy (n) for the piecewise-linear approximation of any monotonic function $\alpha = f(\tau)$ with δ_{max} = const on all the intervals, then the value of the δ_{max} obtained is determined from the expression

$$\delta_{max_i} = \left(\frac{\sqrt[n]{k}-1}{\sqrt[n]{k}+1}\right)\cdot 100, \tag{10}$$

where $k = \alpha_{n+1}/\alpha_1$.

It should be emphasized that the relative values of $\bar{\alpha}_i$ and $\bar{\alpha}_{ai}$ for various δ_{max} in the piecewise-linear approximation of the functions $\bar{\alpha} = \alpha/\alpha_0 = f(\tau)$ can be calculated in advance from relations (4)–(10) and collected in tables whose use considerably accelerates the process of calculating the boundary resistance circuits.

In the process of calculation, the values of α_{ai} on each time interval are determined from the expression

$$\alpha_{a_i} = \bar{\alpha}_{a_i}\alpha_0. \tag{11}$$

The boundaries of the time intervals (τ_{i+1}), at which it is necessary to go over from α_{ai} to α_{ai+1}, are determined graphically from the given functions $\alpha = f(\tau)$. These transitions must be made at times when the function acquires the value α_{i+1}.

Having calculated all the α_i and α_{ai} from (4)–(11), we can construct the piecewise-linear function $\alpha_n = f(\tau)$ which approximates the given function $\alpha = f(\tau)$.

The equations for calculating the boundary resistances can be represented in the form

$$R_{b_i} = \frac{R_{in}}{A}\frac{1}{\alpha_{a_i}}. \tag{12}$$

To reproduce the piecewise-linear function $R_{bn} = f(\tau)$ calculated from (4)–(12), which is an analog of function $\alpha_n = f(\tau)$, it is necessary to set up a boundary resistance circuit in the CRU. The values of the resistances (R'_{bi}) forming the circuit are determined as follows:

$$R'_{b_5} = R_{b_5},$$

$$R'_{b_4} = R_{b_4} - R_{b_5},$$

$$R'_{b_3} = R_{b_3} - R_{b_4} = R_{b_3} - (R'_{b_4} + R'_{b_5}),$$

. . . .

$$R'_{b_i} = R_{b_i} - R_{b_{i+1}} = R_{b_i} - \sum_{i+1}^{n} R'_{b_i}. \tag{13}$$

From these relations it is necessary to calculate the boundary resistance circuits for each boundary point of the region investigated, if in solving problems of nonsteady heat conduction with boundary conditions of the third kind the function $\alpha = f(\tau)$ is approximated with a piecewise-linear one.

Thus, we have demonstrated the possibility of replacing continuous functions $\alpha = f(\tau)$ with piecewise-linear ones with a finite number of intervals of constant α without loss of accuracy in electric-analog solutions of problems of nonsteady heat conduction with time-dependent boundary conditions of the third kind; we have established the feasibility of using boundary resistance circuits to simulate time-dependent heat-transfer coefficients in a controlled resistance unit; and, finally, we have succeeded in developing a method of calculating boundary resistance circuits that minimize the error in solving the problem.

NOTATION

$T_m(x, y, z, \tau)$ is the temperature of the medium as a function of the coordinates and time, deg; $\alpha(x, y, z, \tau)$ is the heat-transfer coefficient as a function of the coordinates and time, $W/m^2 \cdot deg$; λ is the thermal conductivity of the material, $W/m \cdot deg$; c is the specific heat of the material, $W/kg \cdot deg$; γ is the specific weight of the material, kg/m^3; τ is the time, sec; $\bar{\theta}$ is the relative temperature; Bi is the Biot number; α_{ai} is the approximate value of the heat-transfer coefficients assigned upon substituting a piecewise-linear for a continuous law on each time interval $[\tau_i, \tau_{i+1}]$; α_0 is the minimum value of the heat-transfer coefficients during the solution of a problem; δ_{max} is the maximum deviation of the approximate value α_a from the real value on each time interval, in %; R_b is the boundary resistance, ohms; R_{in} is the internal resistance of the part of the network adjacent to the boundary point, ohms; A is a coefficient which depends on the type of boundary point.

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